Written Exam at the Department of Economics Summer 2018

Micro III

Final Exam

August 22, 2018

(2-hour closed book exam)

Answers only in English.

This exam question consists of 3 pages in total (including the current page).

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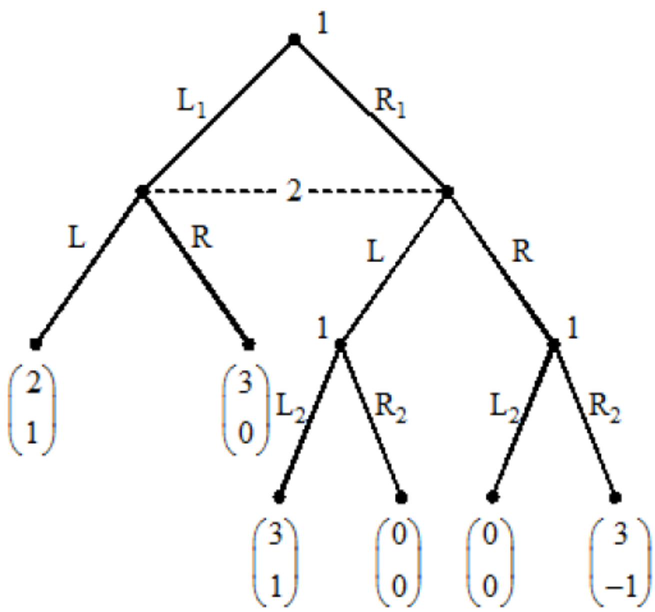
- You cheat at an exam, if during the exam, you:
- Make use of exam aids that are not allowed
- Communicate with or otherwise receive help from other people
- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts
- Or if you otherwise violate the rules that apply to the exam

Solution Guide - Micro III Exam, August 2018

1. Consider the following game G



(a) Briefly explain whether G is a game of perfect or imperfect information (1 sentence).

(b) How many proper subgames are there in G (i.e. not including the game itself)? How many strategies does Player 1 and Player 2 have in this game?

(c) Would your answer to (a) change if we instead considered a game which was identical to G in all respects, expect that Player 1 could not observe whether Player 2 chose L or R? What about your answer to (b)? *Please write just 'Yes' or 'No', for each of these two subquestions.*

(d) Solve for the unique pure strategy subgame perfect Nash equilibrium of G.

(e) Solve for all pure strategy Nash equilibria of G. If there are multiple Nash equilibria, then pick one Nash equilibrium which is not subgame perfect, and explain in words why this is the case (2-3 sentences).

Answer - Question 1

(a) G is a game of imperfect information, since Player 2 has a non-singleton information set.

(b) There are two proper subgames. Player 1 has 8 strategies, and Player 2 has 2 strategies.

(c) No (reason: both players would now have a non-singleton information set). Yes (reason: there would now be no proper subgames, and Player 1 would have 4 strategies).

(d) The unique pure strategy SPNE is $\{(R_1L_2R_2),(L)\}$.

(e) There are two pure strategy NE: $\{(R_1L_2R_2),(L)\}$ and $\{(R_1L_2L_2),(L)\}$. The latter is not subgame perfect, because it specifies that Player 1 should take a suboptimal action, L_2, at one of his decision nodes, which constitutes the bottom-right most subgame. If play actually reached that subgame, then Player 1 would have an incentive to play R_2.

2. Consider a static game F where two firms produce a homogeneous good and compete in quantities. Firm 1 and Firm 2 both produce at zero cost. Let q_i denote the quantity produced by Firm $i \in \{1, 2\}$. Given q_1 and q_2, the market price is $p = 3 - q_1 - q_2$. Both firms choose quantities simultaneously, and maximize profits.

   (a) Solve for the Nash equilibrium of this game. What profits do Firm 1 and Firm 2 earn in equilibrium? What profits would Firm 1 and Firm 2 earn if they instead each produced half of the monopoly quantity (i.e. half of the quantity that maximizes total industry profits)?

   Now consider a dynamic game, with infinite time horizon, where Firm 1 and Firm 2 play the stage game F in periods $t = 1, 2, 3, \ldots$. You can assume that both firms discount future payoffs with factor $\delta \in (0, 1)$.

   (b) Consider a candidate subgame perfect Nash equilibrium where, on the equilibrium path, each firm produces half of the monopoly quantity in each period. Write down trigger strategies for Firm 1 and Firm 2 that could potentially sustain such an equilibrium. Write down an inequality which implicitly defines the values of δ for which neither firm has an incentive to deviate from their equilibrium strategy (you do **not** need to explicitly solve this inequality to isolate δ). Briefly give some intuition as to why the value of the discount factor affects the incentive to deviate (2-3 sentences).

   (c) Now consider a candidate subgame perfect Nash equilibrium where, on the equilibrium path, firms produce the following quantities: in odd periods, $t = 1, 3, 5, \ldots$, Firm 1 produces the monopoly quantity and Firm 2 produces nothing; and in even periods, $t = 2, 4, 6, \ldots$, Firm 2 produces the monopoly quantity and Firm 1 produces nothing. Write down trigger strategies for Firm 1 and Firm 2 that could potentially sustain such an equilibrium. Write down two inequalities which implicitly define the values of δ for which neither firm has an incentive to deviate from their equilibrium strategy (you do **not** need to explicitly solve these inequalities to isolate δ). *Hint 1: think about what is a firm's best reply in a period where the other firm produces zero, and in a period where the other firm produces the monopoly quantity. Hint 2: you may use the fact that* $1 + \delta^2 + \delta^4 + \ldots = \frac{1}{1-\delta^2}$.

   (d) Look back at the inequalities you derived in parts (b) and (c). Can you say something about whether the firms find it easier to sustain collusion if they each produce half the monopoly quantity in each period (as in (b)) or if they take turns each producing the monopoly quantity and zero (as in (c)) (3-4 sentences)? If so, briefly give some intuition (3-4 sentences). *Please attempt this question even if you did not successfully complete the earlier parts.*

Answer - Question 2

(a) Firm i profits are $\pi_i = q_i(3 - q_i - q_j)$ for $i \in \{1, 2\}$. The first-order-condition is $3 - 2q_i - q_j = 0$, yielding a Nash equilibrium of $q_i^* = q_j^* = 1 \equiv q^{n.e.}$ (notice that profit functions are strictly concave, so the second-order condition is always satisfied). Each firm earns equilibrium profits of $\pi^{n.e.} = 1$. Industry profits, given total quantity $q_i + q_j \equiv Q$, are $\pi_i + \pi_j = Q(3 - Q)$. This implies a monopoly quantity of $Q = 3/2$, and a monopoly price of $3/2$. If each firm produced half of the monopoly quantity, $q_i = q_j = 3/4$, then each would earn $\pi^{mon} = 9/8$.

(b) Consider the strategy profile $(Trigger_1, Trigger_2)$, where $Trigger_i$ for firm $i \in \{1, 2\}$ is defined as follows: 'At $t = 1$, choose $q_i = q^{mon} = 3/4$. At $t \geq 2$, choose $q_i = q^{mon} = 3/4$

if $(q_1 = 3/4, q_2 = 3/4)$ were the quantities chosen in all periods $t' \leq t-1$; otherwise choose $q_i = q^{n.e} = 1$.' Off the equilibrium path, both firms always choose the Nash equilibrium quantities of the stage game, which by definition are best responses to one another. Thus, neither firm has an incentive to deviate in subgames that are off the equilibrium path. On the equilibrium path, the relevant condition is $\frac{1}{1-\delta}\pi^{mon} \geq \pi^{dev} + \frac{\delta}{1-\delta}\pi^{n.e}$, where π^{dev} are the profits earned by firm i playing a best response to $q_j = 1$, i.e. setting $q_i = 9/8$, to earn $\pi^{dev} = 81/64$. Direct substitution of $\pi^{mon} = 9/8$, $\pi^{n.e.} = 1$, and $\pi^{dev} = 81/64$ gives:

$$\frac{1}{1-\delta}\frac{9}{8} \geq \frac{81}{64} + \frac{\delta}{1-\delta}1.$$

The strategy profile constitutes a subgame perfect Nash equilibrium for all values of δ satisfying this inequality. Intuitively, when deciding whether to deviate, a firm faces a trade-off: a deviation increases current profits, but decreases payoffs in all future periods, when the firm is punished. The value of the discount factor affects the firm's incentive to deviate because it captures the weight that the firm places on future profits, in relation to current profits, when making this trade off.

(c) Consider the strategy profile $(Trigger_{1'}, Trigger_{2'})$, where $Trigger_{1'}$ for Firm 1 is defined as follows: 'In period 1, choose $q_1 = 3/2$. In any odd period $t > 1$, choose $q_1 = 3/2$ if $(q_1 = 3/2, q_2 = 0)$ were the quantities chosen in all odd periods $t' < t$ and $(q_1 = 0, q_2 = 3/2)$ were the quantities chosen in all even periods $t'' < t$; otherwise choose $q_1 = q^{n.e} = 1$. In any even period $t \geq 2$, choose $q_1 = 0$ if $(q_1 = 3/2, q_2 = 0)$ were the quantities chosen in all odd periods $t' < t$ and $(q_1 = 0, q_2 = 3/2)$ were the quantities chosen in all even periods $t'' < t$; otherwise choose $q_i = q^{n.e} = 1$.' Define $Trigger_{2'}$ for Firm 2 similarly: 'In period 1, choose $q_2 = 0$. In any odd period $t > 1$, choose $q_2 = 0$ if $(q_1 = 3/2, q_2 = 0)$ were the quantities chosen in all odd periods $t' < t$ and $(q_1 = 0, q_2 = 3/2)$ were the quantities chosen in all even periods $t'' < t$; otherwise choose $q_2 = q^{n.e} = 1$. In any even period $t \geq 2$ period, choose $q_2 = 3/2$ if $(q_1 = 3/2, q_2 = 0)$ were the quantities chosen in all odd periods $t' < t$ and $(q_1 = 0, q_2 = 3/2)$ were the quantities chosen in all even periods $t'' < t$; otherwise choose $q_2 = q^{n.e} = 1$.' *Note that a slightly more informal description of the players' strategies is also acceptable, as long as the description is clear.*

To rule out a deviation by Firm 1 in period 1, the relevant inequality can be written as

$$\frac{1}{1-\delta^2}\frac{9}{4} \geq \frac{9}{4} + \frac{\delta}{1-\delta}1.$$

The left-hand side gives Firm 1's equilibrium profits, which consist of total industry monopoly profits $2\pi^{mon} = \frac{9}{4}$ in all even periods, and 0 in all odd periods (here we use Hint 2). To understand the right-hand side, the best Firm 1 can do by deviating is just to marginally adjust its quantity (since it can never earn more than total industry profits), giving approximately $2\pi^{mon} = \frac{9}{4}$. Firm 1 then earns profits of 1 in all later periods where it is punished. An equivalent inequality also captures the incentive to deviate, on the equilibrium path, for Firm 1 in periods $t = 3, 5, \ldots$, and for Firm 2 in periods $t = 2, 4, 6, \ldots$.

To rule out a deviation by Firm 2 in period 1, the relevant inequality can be written as

$$\frac{\delta}{1-\delta^2}\frac{9}{4} \geq \frac{9}{16} + \frac{\delta}{1-\delta}1.$$

The left-hand side gives Firm 2's equilibrium profits, which consist of total industry monopoly profits $2\pi^{mon} = \frac{9}{4}$ in all even periods, and 0 in all odd periods (here we again use Hint 2). The term δ in the numerator reflects the fact that the firm only starts earning positive profits in period 2, rather than period 1. To understand the right-hand side, the best Firm 2 can do by deviating in period 1 is to play the best response to $q_1 = 3/2$, which is $q_2 = 3/4$, and earn profits $9/16$. The firm then earns profits of 1 in all later periods where it is punished. An equivalent inequality also captures the incentive to deviate, on the equilibrium path, for Firm 2 in periods $t = 3, 5, \ldots$, and for Firm 1 in periods $t = 2, 4, 6, \ldots$.

(d) Comparing the inequalities from parts (b) and (c) suggests the answer here is ambiguous. **Firm 2**: In part (c), Firm 2 earns a lower equilibrium a payoff than in part (b), but also earns a lower payoff from deviating in period 1. Intuitively, the fact that Firm 2 must wait until period 2 before earning positive profits increases its incentive to deviate, but the fact that Firm 1 produces the total industry monopoly quantity in period 1 (rather than half that quantity) has the opposite effect, by driving down the price associated with a deviation. **Firm 1**: In part (c), Firm 1 earns a higher equilibrium payoff than in part (b), but also earn a higher payoff from deviating in period 1. Still, the inequalities may suggest that Firm 1 faces a lower overall incentive to deviate in part (c) than in part (b), since the relevant inequality in (c) is satisfied even for very low values of the discount factor. Intuitively, even if the firm was myopic, then it could do no better than earning the total industry monopoly profits in period 1, which is what it earns in equilibrium.

3. A firm is hiring a worker. Workers are characterized by their type θ, which measures their ability. There are two worker types: $\theta \in \{\theta_L, \theta_H\}$. Nature chooses the worker's type, with $\mathbb{P}(\theta = \theta_H) = p$ and $\mathbb{P}(\theta = \theta_L) = 1 - p$. Assume $p \in (0, 1)$. The worker observes his own type, but the firm does not.

   The worker can choose his level of education: $e \in \mathbb{R}^+$. The cost to him of acquiring this education is

$$c_\theta(e) = \frac{e}{\theta}$$

   Education is observed by the firm, who then forms beliefs about the worker's type: $\mu(\theta|e)$. We assume that the marginal productivity of a worker is equal to his ability θ and that the firm is in competition such that it pays the marginal productivity: $w(e) = \mathbb{E}(\theta|e)$. Thus, the payoff to a worker conditional on his type and education is

$$u_\theta(e) = w(e) - c_\theta(e)$$

   Suppose for this exercise that $\theta_H = 4$ and $\theta_L = 2$.

   (a) In a separating equilibrium, the low-ability worker chooses education level e_L and obtains wage $w_L = w(e_L)$. Is it possible that $e_L > 0$? Explain briefly (max. 3 sentences).

   (b) Find a separating pure strategy Perfect Bayesian Equilibrium where the two types choose education levels e_L and e_H, respectively, and the low ability type is indifferent between choosing e_L and e_H. Assume that off the equilibrium path, the firm assigns zero probability to the worker being type θ_H.

   (c) Find a pooling pure strategy Perfect Bayesian Equilibrium in which both types choose education level $\bar{e}$, and the low ability type is indifferent between choosing $e = 0$ and $e = \bar{e}$. Assume that off the equilibrium path, the firm assigns zero probability to the worker being type θ_H. Does the pooling equilibrium you found satisfy Signaling Requirement 6 (*'equilibrium domination'*)? You can show this either graphically or algebraically.

Answer - Question 3

(a) No. Suppose $e_L > 0$. In a separating equilibrium L gets $2 - e_L/2$. For any beliefs, we have $u_L(0) = w(0) \geq \theta_L = 2 > 2 - e_L/2$: a profitable deviation exists.

(b) By assumption, $\mu(\theta|e)$ is equal to 1 is $e = e_H$ and equal to 0 otherwise. Thus, $w(e)$ is equal to 4 when $e = e_H$ and equal to 2 otherwise. We argued above that $e_L = 0$ in equilibrium. Given $w(e)$, $e = e_L = 0$ strictly dominates all $e \neq e_H$ for both types. Hence, only the strategies e_L and e_H need to be considered. To make the low type indifferent:

$$u_L(0) = u_L(e_H) \iff 2 = 4 - \frac{e_H}{2} \iff e_H = 4.$$

   Clearly, the high type prefers $e_H = 4$ as for any $e' \neq e_H$, we have

$$u_H(e') = 2 - \frac{e'}{4} \leq 2 < 4 - \frac{4}{4} = u_H(e_H).$$

   Hence: the specified $w(e)$ and $\mu(\cdot|e)$, together with $e_L = 0$ and $e_H = 4$, constitute a PBE.

(c) $\mu(\theta_H|e)$ is equal to p if $e = \bar{e}$ and equal to 0 otherwise. Thus, $w(e)$ is equal to $p(4) + (1 - p)(2) = 2(1 + p)$ when $e = \bar{e}$ and equal to 2 otherwise. Given $w(e)$, $e = 0$

strictly dominates all $e \neq \overline{e}$ for both types. Hence, only these two strategies need to be considered. To make the low type indifferent:

$$u_L(0) = u_L(\overline{e}) \iff 2 = 2(1+p) - \frac{\overline{e}}{2} \iff \overline{e} = 4p.$$

Clearly, the high type prefers $e = \overline{e} = 4p$ over $e = 0$ as

$$u_H(\overline{e}) = 2(1+p) - \frac{\overline{e}}{4} = 2(1+p) - \frac{4p}{4} = 2 + p \geq 2 = u_H(0)$$

holds. Hence, the specified $w(e)$, $\mu(\cdot|e)$, together with $\overline{e} = 4p$, constitutes a PBE.

Checking SR6: For a low-ability type, choosing $\overline{e}$ always gives a strictly higher payoff than choosing e if

$$2(1+p) - \frac{\overline{e}}{2} > 4 - \frac{e}{2} \iff 2 > 4 - \frac{e}{2} \iff e > 4.$$

For the high-ability type, choosing $\overline{e}$ always gives a strictly higher payoff than choosing e if

$$2(1+p) - \frac{\overline{e}}{4} > 4 - \frac{e}{4} \iff 2 + p > 4 - \frac{e}{4} \iff e > 4(2-p) > 4.$$

Hence, $e \in (4, 4(2-p))$ are equilibrium dominated for L but not for H. SR6: $\mu(\theta_H|e) = 1$ and hence $w(e) = 4$ for $e \in (4, 4(2-p))$. The pooling equilibrium does not satisfy SR6.